Lecture notes on the magnetorotational instability

Anders Johansen
Accretion discs

Intriguing astrophysical questions:

- How do accretion discs accrete? \( \dot{M} = 3\pi \nu_t \Sigma \)
- What is the role of magnetic fields?
- How do planets form in protoplanetary discs?
- How do stars form in AGN discs?
- What is the structure of accretion discs?
Accretion (*)

- Centrifugal balance:
  \[ 0 = \frac{v_\phi^2}{R} - \frac{GM_*}{R^2} \]

- Evolution of surface density:
  \[ \frac{\partial \Sigma}{\partial t} = 3R^{-1} \frac{\partial}{\partial R} \left\{ R^{1/2} \frac{\partial}{\partial R} \left[ \nu \Sigma R^{1/2} \right] \right\} \]

- Mass flux:
  \[ \dot{M} = 3\pi \nu \Sigma \]

- Accretion time-scale:
  \[ t_{\text{acc}} = \frac{R^2}{\nu} \]
Turbulent viscosity

- Molecular viscosity is way too tiny to lead to accretion \((t_{\text{acc}} \sim 10^{20} \text{ s})\)
- Replace \(\nu\) by turbulent viscosity \(\nu_t\)

\[
\langle \rho u_r u_\phi \rangle = \frac{3}{2} \Omega \nu_t^{(\text{kin})} \langle \rho \rangle
\]

\[
-\frac{1}{\mu_0} \langle B_r B_\phi \rangle = \frac{3}{2} \Omega \nu_t^{(\text{mag})} \langle \rho \rangle
\]

- Parameterise our ignorance of \(\nu_t\):

\[
\nu_t^{(\text{kin})} = \alpha_t^{(\text{kin})} c_s H
\]

\[
\nu_t^{(\text{mag})} = \alpha_t^{(\text{mag})} c_s H
\]

- Shakura & Sanuyev (1973) [cited 4244 times]
Magnetorotational instability

What is the source of turbulent stresses in accretion discs?

Balbus & Hawley 1991: Accretion discs penetrated by a weak vertical field are unstable to the magnetorotational instability

1. Shearing sheet frame
2. Stability analysis of magnetised differential rotation
3. Non-linear evolution of the MRI (movies)
4. The role of resistivity and ambipolar diffusion
5. Non-linear hydrodynamical instability
Shearing sheet approximation (*)

Shearing sheet approximation

Magnetic accretion
Anders Johansen

Introduction

Shearing sheet
Stability analysis
Non-linear evolution
Non-ideal MHD effects
Hydro instabilities?
Conclusions
Consider equation of motion of Keplerian disc in cylindrical coordinates $(r, \phi)$:

\[
\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\phi}{r} \frac{\partial u_r}{\partial \phi} - \frac{u_\phi^2}{r} = -\frac{GM_*}{r^2} - \frac{1}{\rho} \frac{\partial P}{\partial r} \\
\frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r u_\phi}{r} = -\frac{1}{r} \frac{\partial P}{\rho \partial \phi}
\]

Take **local, corotating coordinate frame** $(x, y)$ defined through (see e.g. Goldreich & Tremaine 1978)

\[
x = r - r_0 \\
y = r_0[\phi - \Omega(r_0)t]
\]

Need to restate dynamical equations in shearing sheet frame.
Gravity in the shearing sheet

- The gravity from the central star is expanded through \( r = r_0 + x \) and approximated to first order (**): 

\[
g_x = -\frac{GM_*}{(r_0 + x)^2} = -\frac{GM_*}{r_0^2} \left(1 + \frac{2x}{r_0} + \frac{x^2}{r_0^2}\right) 
\approx -\frac{GM_*}{r_0^2} \left(1 - \frac{2x}{r_0}\right)
\]

- Add the centrifugal force \( \left(\frac{GM_*}{r_0^3}\right)(r_0 + x) \) and get tidal force

\[
\frac{\partial u_x}{\partial t} = \ldots + 3\Omega_0^2 x,
\]

where

\[
\Omega_0 = \sqrt{\frac{GM_*}{r_0^3}}
\]

is the Keplerian frequency at \( r = r_0 \).
The induction equation determines the evolution of the magnetic field $B$

For ideal magnetohydrodynamics there are two mathematically equivalent versions of the induction equation (*):

\[
\frac{\partial B}{\partial t} = \nabla \times (u \times B)
\]

\[
\frac{\partial B}{\partial t} + (u \cdot \nabla)B = (B \cdot \nabla)u - B \nabla \cdot u
\]

The latter version is more intuitive: magnetic field is advected by the fluid $[(u \cdot \nabla)B]$ and $B \nabla \cdot u$ and is stretched by shear $[(B \cdot \nabla)u]$. 
Shearing sheet equations

- Equation of motion, continuity equation and induction equation in the shearing sheet approximation:

\[
\begin{align*}
\frac{\partial u_x}{\partial t} + u \cdot \nabla u_x &= 2\Omega_0 u_y + 3\Omega_0^2 x - \frac{1}{\rho} \frac{\partial P}{\partial x} \\
\frac{\partial u_y}{\partial t} + u \cdot \nabla u_y &= -2\Omega_0 u_x - \frac{1}{\rho} \frac{\partial P}{\partial y} \\
\frac{\partial u_z}{\partial t} + u \cdot \nabla u_z &= -\frac{1}{\rho} \frac{\partial P}{\partial z} \\
\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho &= -\rho \nabla \cdot u \\
\frac{\partial B}{\partial t} + u \cdot \nabla B &= B \cdot \nabla u - B \nabla \cdot u
\end{align*}
\]

- Equilibrium flow solution (*):

\[
u_y(x) = -(3/2)\Omega_0 x \quad \text{(linearised Keplerian flow)}
\]
Shearing sheet approximation
Relative to Keplerian flow

- Measure velocities relative to the Keplerian shear flow
  \[ u_y^{(0)} = -(3/2) \Omega_0 x \]

\[
\begin{align*}
\frac{\partial u_x}{\partial t} + \mathbf{u} \cdot \nabla u_x + u_y^{(0)} \frac{\partial u_x}{\partial y} &= 2\Omega_0 u_y + \frac{1}{\rho} (\mathbf{J} \times \mathbf{B})_x - \frac{1}{\rho} \frac{\partial P}{\partial x} \\
\frac{\partial u_y}{\partial t} + \mathbf{u} \cdot \nabla u_y + u_y^{(0)} \frac{\partial u_y}{\partial y} &= -\frac{1}{2} \Omega_0 u_x + \frac{1}{\rho} (\mathbf{J} \times \mathbf{B})_y - \frac{1}{\rho} \frac{\partial P}{\partial y} \\
\frac{\partial u_z}{\partial t} + \mathbf{u} \cdot \nabla u_z + u_y^{(0)} \frac{\partial u_z}{\partial y} &= \frac{1}{\rho} (\mathbf{J} \times \mathbf{B})_z - \frac{1}{\rho} \frac{\partial P}{\partial z} \\
\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + u_y^{(0)} \frac{\partial \rho}{\partial y} &= -\rho \nabla \cdot \mathbf{u} \\
\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} + u_y^{(0)} \frac{\partial \mathbf{B}}{\partial y} &= \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{B} \nabla \cdot \mathbf{u} - \frac{3}{2} \Omega_0 B_x \hat{y}
\end{align*}
\]

- Notice: No centrifugal force, Keplerian advection, Coriolis force changed, introduced explicit magnetic stretching term (*).

- Equilibrium solution: \( \mathbf{u} = 0, \rho = \rho_0, \mathbf{B} = (0, 0, B_0) \)
Stability analysis

Important question:
Do small perturbations to the equilibrium solution oscillate, grow or decay?

Mathematical tool: **linear stability analysis**

1. Choose dynamical variables (e.g. velocity, density, magnetic field)
2. Choose frame of reference and write down dynamical equations
3. Find equilibrium solution
4. Linearise dynamical equations
5. Find out if solutions to linearised equations are oscillatory, damped or growing
We expand our chosen variables in a constant equilibrium part and a fluctuation part:

\[
\begin{align*}
\mathbf{u} &= 0 + \mathbf{u}' \\
\rho &= \rho_0 + \rho' \\
\mathbf{B} &= (0, 0, B_0) + \mathbf{B}' \\
\mathbf{J} &= \mu_0^{-1} \nabla \times \mathbf{B} = \mu_0^{-1} \nabla \times \mathbf{B}' = \mathbf{J}' \\
P &= P_0 + P'
\end{align*}
\]

Then we insert these expressions in the dynamical equations

... and ignore terms that are second order in the fluctuation part
Dynamical equations after insertion

\[
\begin{align*}
\frac{\partial u'_x}{\partial t} + \mathbf{u}' \cdot \nabla u'_x + u'_y \frac{\partial u'_x}{\partial y} &= 2 \Omega_0 u'_y + \frac{1}{\rho_0 + \rho'} (\mathbf{J}' \times \mathbf{B}_0)_x + \frac{1}{\rho_0 + \rho'} (\mathbf{J}' \times \mathbf{B}')_x - \frac{1}{\rho_0 + \rho'} \frac{\partial P'}{\partial x} \\
\frac{\partial u'_y}{\partial t} + \mathbf{u}' \cdot \nabla u'_y + u'_y \frac{\partial u'_y}{\partial y} &= -\frac{1}{2} \Omega_0 u'_x + \frac{1}{\rho_0 + \rho'} (\mathbf{J}' \times \mathbf{B}_0)_y + \frac{1}{\rho_0 + \rho'} (\mathbf{J}' \times \mathbf{B}')_y - \frac{1}{\rho_0 + \rho'} \frac{\partial P'}{\partial y} \\
\frac{\partial u'_z}{\partial t} + \mathbf{u}' \cdot \nabla u'_z + u'_y \frac{\partial u'_z}{\partial y} &= \frac{1}{\rho_0 + \rho'} (\mathbf{J}' \times \mathbf{B}_0)_z + \frac{1}{\rho_0 + \rho'} (\mathbf{J}' \times \mathbf{B}')_z - \frac{1}{\rho_0 + \rho'} \frac{\partial P'}{\partial z} \\
\frac{\partial \rho'}{\partial t} + \mathbf{u}' \cdot \nabla \rho' + u'_y \frac{\partial \rho'}{\partial y} &= -(\rho_0 + \rho') \nabla \cdot \mathbf{u}' \\
\frac{\partial \mathbf{B}'}{\partial t} + \mathbf{u}' \cdot \nabla \mathbf{B}' + u'_y \frac{\partial \mathbf{B}'}{\partial y} &= (\mathbf{B}_0 + \mathbf{B}') \cdot \nabla \mathbf{u}' - (\mathbf{B}_0 + \mathbf{B}') \nabla \cdot \mathbf{u}' - \frac{3}{2} \Omega_0 B'_x \hat{y}
\end{align*}
\]
Let’s make life easy on ourselves and only look at axisymmetric motion (by setting $\partial / \partial y = 0$)
**Axisymmetric equations**

\[
\begin{align*}
\frac{\partial u'_x}{\partial t} &= 2\Omega_0 u'_y + \frac{1}{\rho_0} (J' \times B_0)_x - \frac{1}{\rho_0} \frac{\partial P'}{\partial x} \\
\frac{\partial u'_y}{\partial t} &= -\frac{1}{2} \Omega_0 u'_x + \frac{1}{\rho_0} (J' \times B_0)_y \\
\frac{\partial u'_z}{\partial t} &= \frac{1}{\rho_0} (J' \times B_0)_z - \frac{1}{\rho_0} \frac{\partial P'}{\partial z} \\
\frac{\partial \rho'}{\partial t} &= -\rho_0 \nabla \cdot u' \\
\frac{\partial B'}{\partial t} &= B_0 \cdot \nabla u' - B_0 \nabla \cdot u' - \frac{3}{2} \Omega_0 B'_x \hat{y}
\end{align*}
\]

Let's look in more detail at the Lorentz force term (*):

\[
J \times B = \mu_0^{-1} (\nabla \times B) \times B
\]
\[
= \mu_0^{-1} [B \cdot \nabla B - \nabla (B^2/2)]
\]
\[
= \mu_0^{-1} [B_0 \cdot \nabla B' - \nabla (B_0 \cdot B')]
\]
Expanded Lorentz force

\[
\begin{align*}
\frac{\partial u'_x}{\partial t} &= 2\Omega_0 u'_y + \frac{1}{\mu_0 \rho_0} B_0 \frac{\partial B'_x}{\partial z} - \frac{1}{\mu_0 \rho_0} B_0 \frac{\partial B'_z}{\partial x} - \frac{1}{\rho_0} \frac{\partial P'}{\partial x} \\
\frac{\partial u'_y}{\partial t} &= -\frac{1}{2} \Omega_0 u'_x + \frac{1}{\mu_0 \rho_0} B_0 \frac{\partial B'_y}{\partial z} \\
\frac{\partial u'_z}{\partial t} &= -\frac{1}{\rho_0} \frac{\partial P'}{\partial z} \\
\frac{\partial \rho'}{\partial t} &= -\rho_0 \left( \frac{\partial u'_x}{\partial x} + \frac{\partial u'_z}{\partial z} \right) \\
\frac{\partial B'}{\partial t} &= B_0 \frac{\partial u'}{\partial z} - B_0 \left( \frac{\partial u'_x}{\partial x} + \frac{\partial u'_z}{\partial z} \right) \hat{z} - \frac{3}{2} \Omega_0 B'_x \hat{y}
\end{align*}
\]

- Lorentz force has no effect on vertical component of velocity
- Let’s make life even simpler: we ignore radial variation, considering only vertical variation
Ignoring radial variation

\[ \frac{\partial u'_x}{\partial t} = 2\Omega_0 u'_y + \frac{1}{\mu_0 \rho_0} B_0 \frac{\partial B'_x}{\partial z} \]

\[ \frac{\partial u'_y}{\partial t} = -\frac{1}{2} \Omega_0 u'_x + \frac{1}{\mu_0 \rho_0} B_0 \frac{\partial B'_y}{\partial z} \]

\[ \frac{\partial u'_z}{\partial t} = -\frac{1}{\rho_0} \frac{\partial P'}{\partial z} \]

\[ \frac{\partial \rho'}{\partial t} = -\rho_0 \frac{\partial u'_z}{\partial z} \]

\[ \frac{\partial B'}{\partial t} = B_0 \frac{\partial u'}{\partial z} - B_0 \frac{\partial u'_z}{\partial z} \hat{z} - \frac{3}{2} \Omega_0 B'_x \hat{y} \]

- Evident that \( u'_z \) and \( \rho' \) decouple from the equation system
No evolution of $u'_z$ and $\rho'$

\[
\begin{align*}
\frac{\partial u'_x}{\partial t} &= 2\Omega_0 u'_y + \frac{1}{\mu_0 \rho_0} B_0 \frac{\partial B'_x}{\partial z} \\
\frac{\partial u'_y}{\partial t} &= -\frac{1}{2} \Omega_0 u'_x + \frac{1}{\mu_0 \rho_0} B_0 \frac{\partial B'_y}{\partial z} \\
\frac{\partial B'_x}{\partial t} &= B_0 \frac{\partial u'_x}{\partial z} \\
\frac{\partial B'_y}{\partial t} &= B_0 \frac{\partial u'_y}{\partial z} - \frac{3}{2} \Omega_0 B'_x \\
\frac{\partial B'_z}{\partial t} &= B_0 \frac{\partial u'_z}{\partial z} - B_0 \frac{\partial u'_z}{\partial z}
\end{align*}
\]

- Vertical magnetic field does not evolve
Final set of evolution equations

- Dynamical equations for evolution of small perturbations to accretion disc threaded by vertical magnetic field $B_0$

\[
\begin{align*}
\frac{\partial u'_x}{\partial t} & = 2\Omega_0 u'_y + \frac{1}{\mu_0 \rho_0} B_0 \frac{\partial B'_x}{\partial z} \\
\frac{\partial u'_y}{\partial t} & = -\frac{1}{2} \Omega_0 u'_x + \frac{1}{\mu_0 \rho_0} B_0 \frac{\partial B'_y}{\partial z} \\
\frac{\partial B'_x}{\partial t} & = B_0 \frac{\partial u'_x}{\partial z} \\
\frac{\partial B'_y}{\partial t} & = B_0 \frac{\partial u'_y}{\partial z} - \frac{3}{2} \Omega_0 B'_x
\end{align*}
\]

- After much fiddling we are left with only a few terms:
  - Coriolis force
  - Magnetic tension force
  - Magnetic stretching
Final set of evolution equations

\[
\frac{\partial u'_{x}}{\partial t} = 2\Omega_0 u'_{y} + \frac{1}{\mu_0 \rho_0} B_0 \frac{\partial B'_{x}}{\partial z}
\]
\[
\frac{\partial u'_{y}}{\partial t} = -\frac{1}{2} \Omega_0 u'_{x} + \frac{1}{\mu_0 \rho_0} B_0 \frac{\partial B'_{y}}{\partial z}
\]
\[
\frac{\partial B'_{x}}{\partial t} = B_0 \frac{\partial u'_{x}}{\partial z}
\]
\[
\frac{\partial B'_{y}}{\partial t} = B_0 \frac{\partial u'_{y}}{\partial z} - \frac{3}{2} \Omega_0 B'_{x}
\]

After much fiddling we are left with only a few terms:

- **Coriolis force**
- **Magnetic tension force**
- **Magnetic stretching**
Solution

- **Homogeneous system of linear partial differential equations**
- Solutions are known to be of the type

  \[ q(z, t) = \hat{q} \exp[i(k_z z - \omega t)] \]

Here \( q = [u'_x(t), u'_y(t), B'_x(t), B'_y(t)] \).

- Can reformulate the dynamical equations in terms of \( \hat{q} \)
- We are looking now for dispersion relation which gives \( \omega \) as a function of wavenumber \( k_z \). Three possibilities (*)

  - \( \omega^2 > 0 \): Oscillatory solution (e.g. sound wave)
  - \( \omega^2 < 0, \text{Im}(\omega) > 0 \): Growing solution
  - \( \omega^2 < 0, \text{Im}(\omega) < 0 \): Damped solution
Insert exponential solution

- Inserting exponential solution

\[ q(z, t) = \hat{q} \exp[i(k_z z - \omega t)] = \hat{q} \exp[i k_z z] \exp[-i \omega t], \]

we may replace \( \partial/\partial t \rightarrow -i \omega \cdot \) and \( \partial/\partial z \rightarrow i k_z \cdot \)

\[
\begin{align*}
-i \omega \hat{u}_x &= 2 \Omega_0 \hat{u}_y + \frac{1}{\mu_0 \rho_0} B_0 i k_z \hat{B}_x \\
-i \omega \hat{u}_y &= -\frac{1}{2} \Omega_0 \hat{u}_x + \frac{1}{\mu_0 \rho_0} B_0 i k_z B_y \\
-i \omega \hat{B}_x &= B_0 i k_z \hat{u}_x \\
-i \omega \hat{B}_y &= B_0 i k_z \hat{u}_y - \frac{3}{2} \Omega_0 \hat{B}_x
\end{align*}
\]

- Turned partial differential equation into algebraic equation
- Trivial (boring) solution: \( \hat{u}_x = \hat{u}_y = \hat{B}_x = \hat{B}_y = 0 \)
Now as a matrix

- Write algebraic equation as matrix:

\[
\begin{pmatrix}
  i\omega & 2\Omega_0 & \frac{1}{\mu_0\rho_0} B_0 i k_z & 0 \\
  -\frac{1}{2} \Omega_0 & i\omega & 0 & \frac{1}{\mu_0\rho_0} B_0 i k_z \\
  B_0 i k_z & 0 & i\omega & 0 \\
  0 & B_0 i k_z & -\frac{3}{2} \Omega_0 & i\omega \\
\end{pmatrix}
\begin{pmatrix}
  \hat{u}_x \\
  \hat{u}_y \\
  \hat{B}_x \\
  \hat{B}_y \\
\end{pmatrix} = 0
\]

- We have non-trivial solutions only if determinant is zero:

\[
0 = \omega^4 - \omega^2(2v_A^2 k_z^2 + \Omega^2) + v_A^2 k_z^2 (v_A^2 k_z^2 - 3\Omega^2)
\]

- Definition of Alfvén speed: \( v_A^2 = B_0^2 / (\mu_0\rho_0) \)
This is a biquadratic equation, or a quadratic equation in $\omega^2$. The two solutions are $\omega_1^2$ and $\omega_2^2$. The equation can be rewritten

$$\omega^4 - \omega^2 (\omega_1^2 + \omega_2^2) + \omega_1^2 \omega_2^2 = 0$$

One of the two solutions is negative when

$$k^2 v_A^2 < 3\Omega^2$$

The solution is (*)

$$\omega_{1,2}^2 = v_A^2 k_z^2 + \frac{1}{2} \Omega^2 \pm \sqrt{4v_A^2 k_z^2 \Omega^2 + \frac{1}{4} \Omega^4}$$

Means that $\omega$ is either purely real or purely complex
Plot of $\omega^2$

- Plot of $\omega^2$ versus $k_z$ for $v_A = \Omega = 1$:

- Instability between $k_z = 0$ and $k_z = \sqrt{3} \Omega / v_A$
- Lowest value of $\omega^2$ for $k_z = \sqrt{15/16} \Omega / v_A$
- Here growth rate is $\gamma = \text{Im}(\omega) = (3/4) \Omega$
Summary of MRI

Most unstable wavelength: $\lambda = 2\pi \sqrt{\frac{16}{15}} \frac{v_A}{\Omega}$

Growth rate: $\gamma = (3/4)\Omega$.

- Growth rate is independent of background magnetic field, so growth occurs even for weak fields
- Fastest growing wavelength proportional to $B_0$
- Growth is not possible for very strong fields (unstable wavelengths are too large for the disc)
- Growth may also be hindered by resistivity if ionisation fraction is not high enough
Two masses on a spring

Balbus & Hawley (1991):
A weak vertical magnetic field renders Keplerian shear flows linearly unstable to the magnetorotational instability

MRI transports angular momentum through Maxwell and Reynolds stresses if degree of ionisation high enough

Based on earlier ideas by Velikhov and by Chandrasekhar
Two masses on a spring

Magnetic accretion
Anders Johansen

Introduction
Shearing sheet
Stability analysis
Non-linear evolution
Non-ideal MHD effects
Hydro instabilities?
Conclusions

Image courtesy of W. Lyra
5.3. The Magnetorotational Instability: A Brief History

The understanding that the characteristic accretion disk combination of Keplerian rotation and a magnetic field is highly unstable emerged very belatedly, only a little more than a decade ago. Why this is so is an interesting question, given that disk stability research was already some 30 years old at the time, and modern accretion disk theory had been around for more than 20 years. The MRI calculation is not a difficult one, and work on the topic was initiated by no less a personage than Chandrasekhar (1953). How could such an important instability have been overlooked for so long? In fact, it was not. But neither was it well understood. The early studies by Chandrasekhar were extremely formal. Indeed, his 1953 paper was not at all astrophysical in context; it was a theoretical Couette flow analysis. Chandrasekhar limited his work on hydrodynamic and hydromagnetic stability to those configurations in which a static equilibrium could be defined and analyzed globally. Yet the lack of a coherent physical explanation of the instability hampered its understanding, making it difficult to perceive both its local character and widespread generality.
Some history

Recently proposed MRI laboratory experiments (Ji, Goodman & Kageyama 2001; Noguchi et al. 2002; Rüdiger & Shalybkov 2002) have led to a detailed reexamination of Chandrasekhar's analysis of dissipative Couette flow, and surprisingly, there appears to have been a rather straightforward oversight (Goodman & Ji 2002). Magnetized shear flow will of course draw out an azimuthal field from any radial component that happens to be present. In Chandrasekhar's discussion, the term responsible for this behavior was dropped from the induction equation on the grounds of a small magnetic Prandtl number approximation. This was most unfortunate because the process represented by this dropped term need not vanish in this limit; indeed, it is critical for the MRI to function.
Some history

How widespread this misapprehension became is difficult to say. [The error made its way into Chandrasekhars classic text (1961).] Much of the fluid and stellar community was aware of the instability, however, and of its curious behavior of ostensibly changing the Rayleigh criterion discontinuously. Noteworthy studies include those of Velikhov (1959) (the first derivation of a dissipationless angular velocity stability criterion for Couette profiles), Chandrasekhar (1960) (a generalization of Velikhov's result), Newcomb (1962) (a variational approach), Fricke (1969) (stellar differential rotation), Acheson & Hide (1972) (geophysical applications), and Acheson & Gibbons (1978) (more general stellar applications). Yet the robustness and the very general tendency of subthermal magnetic fields of any geometry to destabilize flows was never explicitly discussed. Indeed, the common wisdom was that magnetic fields were thought to be a stabilizing influence, and they were certainly a computational nuisance. It is telling that there are no references in the accretion disk literature to either Velikhov (1959) or Chandrasekhar (1960) prior to 1991.
The key conceptual point, which really has been grasped only in the last decade, is that the limit $B \to 0$ retains the most salient behavioral features of a magnetized fluid and can be profitably investigated. One should think of an accretion disk not as a rotating fluid with a magnetic field, but as a magnetized fluid with rotation. By tethering fluid elements, magnetic fields impart dynamical significance to free energy gradients, which otherwise are felt only through their more ghostly diffusive presence. The stability properties of even a barely magnetized fluid are qualitatively different from those of a nonmagnetized fluid - a result that holds whether the fluid is rotating or not.

From Balbus (2003, Annual Review of Astronomy & Astrophysics)
Evolution from noise
Evolution from noise
To model the non-linear evolution of the MRI, we need to solve the full set of dynamical equations numerically in 3-D.
Shearing sheet dynamical equations

- Measure velocities relative to the Keplerian shear flow $u_y^{(0)} = -(3/2)\Omega_0 x$:

\[
\begin{align*}
\frac{\partial u_x}{\partial t} + \mathbf{u} \cdot \nabla u_x + u_y^{(0)} \frac{\partial u_x}{\partial y} &= 2\Omega_0 u_y + \frac{1}{\rho} (\mathbf{J} \times \mathbf{B})_x - \frac{1}{\rho} \frac{\partial P}{\partial x} \\
\frac{\partial u_y}{\partial t} + \mathbf{u} \cdot \nabla u_y + u_y^{(0)} \frac{\partial u_y}{\partial y} &= -\frac{1}{2} \Omega_0 u_x + \frac{1}{\rho} (\mathbf{J} \times \mathbf{B})_y - \frac{1}{\rho} \frac{\partial P}{\partial y} \\
\frac{\partial u_z}{\partial t} + \mathbf{u} \cdot \nabla u_z + u_y^{(0)} \frac{\partial u_z}{\partial y} &= \frac{1}{\rho} (\mathbf{J} \times \mathbf{B})_z - \frac{1}{\rho} \frac{\partial P}{\partial z} \\
\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + u_y^{(0)} \frac{\partial \rho}{\partial y} &= -\rho \nabla \cdot \mathbf{u} \\
\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} + u_y^{(0)} \frac{\partial \mathbf{B}}{\partial y} &= \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{B} \nabla \cdot \mathbf{u} - \frac{3}{2} \Omega_0 B_x \hat{y}
\end{align*}
\]

- Need computer code to solve numerical simulations
- Several good codes can be downloaded for free
Illustration of *non-linear evolution* of MRI from Sano et al. (2004):

- **Initial patch of vertical field**
- **Slow magnetosonic wave** is linearly unstable
- **Makes entire box turbulent**
- **Maxwell and Reynolds stress gives** $\alpha$-viscosity
- **No significant dependence on boundary conditions**
Maxwell stresses and Reynolds stresses

- Both Maxwell and Reynolds stress are positive – angular momentum transported out while mass is transported in.
- Maxwell stress is around four times larger than Reynolds stress.
- Turbulent viscosity $\alpha \approx 0.001$.
Dependence on background magnetic field

- Plot of turbulent viscosity as a function of imposed field strength ($B_0$):

- Exponential rise of $\alpha$ with $B_0$
Resistivity

Electrical resistivity is a measure of how strongly a material opposes the flow of electric current. A low resistivity indicates a material that readily allows the movement of electrical charge.

- So far we have looked at the magnetorotational instability in the limit of perfect coupling between the magnetic field and the velocity field.
- In astrophysical context resistivity is related to the ionisation fraction of the gas.
Maxwell equations revisited

- Faraday’s law of induction for evolution of magnetic field $\mathbf{B}$:
  \[ \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \]

- Ampère’s law for evolution of electric field $\mathbf{E}$:
  \[ \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \mu_0 \mathbf{J} \]

- Ohm’s law for the current density $\mathbf{J}$:
  \[ \mathbf{J} = \sigma_e (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \]

Here $\sigma_e$ is the electric conductivity.
Induction equation with resistivity

- Insert Ohm’s law into Faraday’s law:

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \mathbf{u} \times \mathbf{B} - \frac{1}{\sigma_e} \mathbf{J} \right)
\]

\[
= \nabla \times \left( \mathbf{u} \times \mathbf{B} - \frac{1}{\mu_0 \sigma_e} \nabla \times \mathbf{B} \right)
\]

\[
= \nabla \times (\mathbf{u} \times \mathbf{B} - \eta \nabla \times \mathbf{B})
\]

- Here \( \eta \equiv \frac{1}{\mu_0 \sigma_e} \) is the resistivity.

Use vector calculus identity:

\[
\nabla \times \nabla \times \mathbf{B} = \nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}
\]

- Induction equation including resistivity:

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}
\]
Damping by resistivity

- Induction equation including resistivity:
  \[
  \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}
  \]

- For simplicity, let’s set \(\mathbf{u} = 0\) to get:
  \[
  \frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}
  \]

- Insert exponential solution \(\mathbf{B}(t) = \hat{\mathbf{B}} \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]\): (\(*\))
  \[
  -i\omega \hat{\mathbf{B}} = -\eta k^2 \hat{\mathbf{B}}
  \]

- The frequency \(\omega\) is negative complex:
  \[
  \omega = -i\eta k^2
  \]

- Resistivity leads to damping of the magnetic field:
  \[
  \mathbf{B}(t) = \hat{\mathbf{B}} \exp(-\eta k^2 t) \exp[i(\mathbf{k} \cdot \mathbf{x})]
  \]
Magnetic accretion
Anders Johansen
Introduction
Shearing sheet
Stability analysis
Non-linear evolution
Non-ideal MHD effects
Hydro instabilities?
Conclusions

Damping of MRI

- Instability between \( k_z = 0 \) and \( k_z = \sqrt{3} \Omega / v_A \)
- Lowest value of \( \omega^2 \) for \( k_{BH} = \sqrt{15/16} \Omega / v_A \)
- Here \( \gamma_+ = (3/4) \Omega \)

- Resistive damping of Balbus-Hawley wavenumber:

\[
\gamma_- = -\eta k_{BH}^2 = \frac{15}{16} \eta \frac{\Omega^2}{v_A^2}
\]

- The magnetorotational instability is suppressed for (*)

\[
\eta \gtrsim \frac{v_A^2}{\Omega}
\]
The resistivity of the electrons is given in c.g.s. units by

$$\eta = \frac{c^2}{4\pi \sigma_e}$$

where $c$ is the speed of light and $\sigma_e$ is the electrical conductivity. In turn the conductivity is given by

$$\sigma_e = \frac{n_{e}e^{2}}{m_{e}\nu}$$

Here $n_e$ is the number density of electrons, $e$ is the electron charge, $m_e$ is the electron mass, and $\nu$ is the collision frequency of electrons with neutrals. The momentum rate coefficient $\langle \sigma \nu \rangle = \nu/n_{n}$ for transfer of momentum from electrons to neutrals is given by

$$\langle \sigma \nu \rangle = 8.3 \times 10^{-10} T^{1/2} \text{cm}^3 \text{s}^{-1}$$

This finally yields

$$\eta = 230 \left( \frac{n_{n}}{n_{e}} \right) T^{1/2} \text{cm}^2 \text{s}^{-1}$$
Ionisation fraction

- Resistivity of electrons due to collisions with neutrals:
  \[ \eta = 230 \left( \frac{n_n}{n_e} \right) T^{1/2} \text{ cm}^2 \text{ s}^{-1} \]

- Damping rate due to resistivity:
  \[ \gamma_- = -\eta k_z^2 \]

- Growth rate of the MRI at \( k_z \approx \Omega/\nu_A \):
  \[ \gamma_+ = (3/4) \Omega \]

- Growth only if ionisation fraction fulfills:
  \[ \frac{n_e}{n_n} \gg 6.6 \times 10^{-14} \beta \left( \frac{T}{100 \text{ K}} \right)^{-1/2} \left( \frac{P}{\text{ years}} \right)^{-1} \]

  Here \( \beta = P_{\text{thermal}}/P_{\text{mag}} \).
Cosmic ray ionisation

Consider flux of cosmic rays \( F \) (particles per area per unit time). The reduction of the flux is controlled by the equation

\[
\frac{dF}{dz} = -\kappa \rho(z) F(z)
\]

where \( \kappa \) is the opacity and \( \rho \) is the \( z \)-dependent mass density. The solution is

\[
F_\downarrow(z) = F_\infty \exp\left[-\kappa \Sigma_\uparrow(z)\right] \\
F_\uparrow(z) = F_\infty \exp\left[-\kappa \Sigma_\downarrow(z)\right]
\]

Here \( \Sigma_\uparrow(z) = \int_z^\infty \rho(z) dz \) is the column density of gas above the given point, while \( \Sigma_\downarrow(z) = \int_{-\infty}^{z} \rho(z) dz \) is the column density below. We have \( \Sigma = \Sigma_\uparrow(z) + \Sigma_\downarrow(z) \) at all \( z \).
Cosmic ray ionisation

Introducing the ionisation rate $\zeta(z)$ we get

$$\zeta(z) = \frac{\zeta_{CR}}{2} \left\{ \exp\left[ -\Sigma^{\downarrow}(z)/\Sigma_{CR} \right] + \exp\left[ -\Sigma^{\uparrow}(z)/\Sigma_{CR} \right] \right\}$$

Here $\zeta_{CR}$ is the ionisation rate by cosmic rays in interstellar space and $\Sigma_{CR} = 1/\kappa$ is the penetration column density of cosmic rays. Free electrons are lost as they collide with dust grains. This yields the rate equation

$$\frac{\partial n_{e}}{\partial t} = \zeta n_{n} - n_{e} n_{\bullet} \langle \sigma v \rangle_{e,\bullet}$$

for electron number density $n_{e}$. Here $\zeta = \kappa m_{n} F$ is the ionisation rate. The equilibrium electron density fraction is

$$\frac{n_{e}}{n_{n}} = \frac{\zeta}{n_{\bullet} \langle \sigma v \rangle_{e,\bullet}}$$
Cosmic ray ionisation

- Ionisation fraction due to grains:

\[ x = \frac{\zeta}{n_\bullet \langle \sigma v \rangle_{e,\bullet}} \]

Cosmic ray flux \( \zeta \approx 10^{-17} \text{ s}^{-1} \) per neutral hydrogen.

- Ionisation fraction due to recombination on ions: (*)

\[ x = \left( \frac{\zeta}{\beta n_n} \right)^{1/2} = 10^{-12} \left( \frac{\zeta(z)}{10^{-17} \text{ s}^{-1}} \right)^{1/2} \left( \frac{T}{100 \text{ K}} \right)^{1/4} \left( \frac{n_n}{10^{13} \text{ cm}^{-3}} \right)^{1/2} \]

Recombination coefficient \( \beta = 8.7 \times 10^{-6} T^{-1/2} \text{ cm}^3 \text{ s}^{-1} \).

This is an lower limit to the actual ionisation fraction (Gammie 1996).
Dead zone

- Image from Sano et al. (2004)
- Ionisation model taking into account dust grains
- Protoplanetary discs have insufficient ionisation fraction in the inner part
- “Dead zone” (Gammie 1996)
Dependence on dust density

- Dead zone vanishes as dust grains grow larger
Layered accretion

Fleming & Stone (2003):
Numerical simulation of MRI with resistivity profile

- Results in layered accretion
- Figure from Oishi et al. (2007)
- Mid-plane magnetic Reynolds number $Re_M$ varied from 3 to $\infty$ (*)
Activity in the dead zone

- Maxwell stresses disappear in the dead zone, but Reynolds stresses remain
Ambipolar diffusion

- Ions need to transfer momentum to the neutrals in order for MRI to really affect the disc
- The equation of motion for neutrals and ions coupled by collision drag force are

\[
\frac{\partial v_n}{\partial t} + v_n \cdot \nabla v_n = \nabla \Phi - \frac{1}{\rho_n} \nabla P_n - \gamma \rho_i (v_n - v_i)
\]

\[
\frac{\partial v_i}{\partial t} + v_i \cdot \nabla v_i = \nabla \Phi - \frac{1}{\rho_i} \nabla P_i + \frac{(\nabla \times B) \times B}{\mu_0 \rho_i} - \gamma \rho_n (v_i - v_n)
\]

- The ion-neutrals coupling time-scale is

\[
t_{AD} = \frac{1}{\gamma \rho_i}
\]

- Drag coefficient \( \gamma = 3.5 \times 10^{13} \text{ cm}^3 \text{s}^{-1} \text{g}^{-1} \)
In the strong coupling limit we set

\[ \mu_0 \gamma \rho_n \rho_i (v_i - v_n) = (\nabla \times \mathbf{B}) \times \mathbf{B} \]

and obtain

\[ \frac{\partial v_n}{\partial t} + v_n \cdot \nabla v_n = \nabla \Phi + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{\mu_0 \rho_n} - \frac{1}{\rho_n} \nabla P_n \]

Require \( \gamma \rho_i > \Omega_0 \) for MRI to operate efficiently (Blaes & Balbus 1994)

A neutral molecule must collide with at least one ion per orbit (MRI growth time-scale)
Rayleigh criterion

- General evolution equations for $\Omega(R) \propto R^{-q}$:

$$
\begin{align*}
\frac{\partial u_x}{\partial t} + u \cdot \nabla u_x + u_y^{(0)} \frac{\partial u_x}{\partial y} &= 2\Omega_0 u_y - \frac{1}{\rho} \frac{\partial P}{\partial x} \\
\frac{\partial u_y}{\partial t} + u \cdot \nabla u_y + u_y^{(0)} \frac{\partial u_y}{\partial y} &= -(2 - q)\Omega_0 u_x - \frac{1}{\rho} \frac{\partial P}{\partial y} \\
\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho + u_y^{(0)} \frac{\partial \rho}{\partial y} &= -\rho \nabla \cdot u
\end{align*}
$$

- Let’s ignore pressure and $\partial/\partial y$ – effectively ignoring density waves and inertial waves.
Rayleigh criterion

- General evolution equations for $\Omega(R) \propto R^{-q}$:

$$
\frac{\partial u_x}{\partial t} + \mathbf{u} \cdot \nabla u_x = 2\Omega_0 u_y \\
\frac{\partial u_y}{\partial t} + \mathbf{u} \cdot \nabla u_y = -(2-q)\Omega_0 u_x
$$

- Linearised

$$
\frac{\partial u'_x}{\partial t} = 2\Omega_0 u'_y \\
\frac{\partial u'_y}{\partial t} = -(2-q)\Omega_0 u'_x
$$
Rayleigh criterion

- Insert \( q = \hat{q} \exp[-i \omega t] \):
  
  \[
  \begin{align*}
  -i \omega \hat{u}'_x &= 2 \Omega_0 \hat{u}'_y \nonumber \\
  -i \omega \hat{u}'_y &= -(2 - q) \Omega_0 \hat{u}'_x 
  \end{align*}
  \]

- As a matrix:
  
  \[
  \begin{pmatrix}
  i \omega & 2 \Omega_0 \\
  -(2 - q) \Omega_0 & i \omega
  \end{pmatrix}
  \begin{pmatrix}
  \hat{u}'_x \\
  \hat{u}'_y
  \end{pmatrix} = 0
  \]

- Determinant:
  
  \[
  -\omega^2 + (2 - q) \Omega_0^2 = 0
  \]

- Stability for \( q < 2 \) (Rayleigh stability criterion)
- Keplerian shear flows are linearly stable in absence of a magnetic field
Non-linear instabilities

- Linear instability analysis can only find instabilities that operate even for infinitesimal perturbations
- What about non-linear instabilities? (*)
- Coriolis force stabilises, but becomes ineffective at small scales
- Computer simulations currently do not have high enough resolution to probe $Re \sim 10^6$ or higher
- Turn to laboratory experiments instead
Hydrodynamic turbulence cannot transport angular momentum effectively in astrophysical disks

Hantao Ji¹, Michael Burin¹, Ethan Schartman¹ & Jeremy Goodman¹

The most efficient energy sources known in the Universe are accretion disks. Those around black holes convert 5–40 per cent of rest-mass energy to radiation. Like water circling a drain, inflowing mass must lose angular momentum, presumably by vigorous turbulence in disks, which are essentially inviscid². The origin of the turbulence is unclear. Hot disks of electrically conducting plasma can become turbulent by way of the linear magnetorotational instability³. Cool disks, such as the planet-forming disks of protostars, may be too poorly ionized for the magnetorotational instability to occur, and therefore essentially unmagnetized and linearly stable. Nonlinear hydrodynamic instability often occurs in linearly stable flows (for example, pipe flows) at sufficiently large Reynolds numbers. Although planet-forming disks have extreme Reynolds numbers, keplerian rotation enhances their linear hydrodynamic stability, so the question of whether they can be turbulent and thereby transport angular momentum effectively is controversial⁴⁵. Here we report a laboratory experiment, demonstrating that non-magnetic quasi-keplerian flows at Reynolds numbers up to millions are essentially steady. Scaled to accretion disks, rates of angular momentum transport lie far below astrophysical requirements. By ruling out purely hydrodynamic turbulence, our results indirectly support the magnetorotational instability as the likely cause of turbulence, even in cool disks.

Our experiments involved a novel Taylor–Couette apparatus⁶. The rotating liquid (water or a water/glycerol mixture) is confined between two concentric cylinders of radii \( r_1, r_2 \) \((r_2 > r_1)\) and height \( h \). The angular velocity of the fluid is controlled by that of the cylinders,
No significant Reynolds stress up to a Reynolds number of $2 \times 10^6$.
Asterisks show effect of endcaps.
If non-linear instabilities set in at $Re > 10^6$, one can show that results in $\alpha < 10^{-6}$.
“Although it has been suggested that complications such as vertical or radial stratification may yet lead to essentially linear non-axisymmetric hydrodynamic instabilities, our belief is that such non-axisymmetric linear instabilities depend upon radial boundaries and hence are not generally important in thin disks. If this is correct, then by default, the magnetorotational instability appears to be the only plausible source of accretion disk turbulence.”
Meteoritic evidence for magnetic fields

Magnetic accretion
Anders Johansen

Introduction
Shearing sheet
Stability analysis
Non-linear evolution
Non-ideal MHD effects
Hydro instabilities?
Conclusions

Meteoritic evidence for magnetic fields

- **Natural Remanent Magnetism** found in most meteorites
  - Levy & Sonett (1978)
- Meteorites have remnant magnetisation of $0.01 - 10 \text{ G}$
- Frozen in during cooling past Curie temperature
- Parent bodies of undifferentiated meteorites very unlikely to have magnetic field
- Magnetic field internal to disc, dragged in from envelope or originating in the young sun?

**Murchison meteorite**

Levy & Sonett (1978)

Meteorites have remnant magnetisation of $0.01 - 10 \text{ G}$

Frozen in during cooling past Curie temperature

Parent bodies of undifferentiated meteorites very unlikely to have magnetic field

Magnetic field internal to disc, dragged in from envelope or originating in the young sun?
Concluding remarks

- Keplerian flows are linearly unstable to small perturbations in the presence of a vertical magnetic field.
- The growth rate of the instability is \((3/4)\) times the Keplerian time-scale.
- The most unstable wavelength is proportional to the strength of the background magnetic field.
- Non-linear solution is a turbulent state.
- Turbulent stresses cause discs to accrete – the MRI puts the accretion in accretion discs.
- In cold discs, the resistivity may be too high for the magnetorotational instability to operate.
- Neutrals need to collide with at least one ion per orbit.
- Stability analysis is an extremely strong mathematical tool.